As in Sect. 2, conditions (5.11) can be reduced to the form

$$a + \varepsilon^2 > 0, \quad \nu_0 > \nu_2^\circ \quad (2a\nu_2^\circ = b + \sqrt{b^2 - 4ac} + (1 + \nu_0)\varepsilon^2)$$

A similar situation obtains in case (4.1).

Thus, to within terms of the order ε^2 the geometric interpretation of the set of steady motions given in Sects. 1 and 3 and also stability conditions (2.7) and (4.3) for cases (1.1) and (4.1), respectively, are also valid for the unrestricted formulation of the problem. Conditions (2.7) and (4.3) are the stability conditions with respect to γ_i, β_i, R, $\varkappa$, $\gamma_i^{\cdot}$, $\beta_i^{\cdot}$, $R^{\cdot}$, $\varkappa^{\cdot}$, $\sigma^{\cdot}$ with allowance for the perturbability of the orbit.

6. Stability conditions (2.7) and (4.3) remain valid in the case where the satellite contains, in addition to the rotors, cavities completely filled with liquid [7].

The author is grateful to V. V. Rumiantsev for his comments and useful suggestions.

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DYNAMICS OF AN ELECTROMAGNETIC TRIGGER REGULATOR WITH TWO PULSES PER PERIOD

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The dynamics of an electromagnetically driven electromechanical trigger regulator with two pulses per period is considered. The nonlinear third-order differential equation is investigated by the method of point transformations. The decomposition of the parameter space into domains whose points correspond to various qualitative structures of the phase space is established. The domains of existence of several stable periodic motions in the parameter space are isolated.

1. The equation of motion. The motion of an electromagnetically driven electromechanical trigger regulator with two pulses per period can be investigated by means of the model described in [1]. The contact device in the modified model closes the electrical circuit both with left-to-right and with reverse motion of the oscillator (i.e. there are two pulse zones situated symmetrically with respect to the position of static equilibrium).

The equations of motion of the above dynamic system in dimensionless variables are

$$\begin{aligned} &x^{\cdot\cdot} + x = (y^2 - r)\frac{x^{\cdot}}{|x^{\cdot}|} && \text{for } x^{\cdot} \geqslant 0,\ |x + b + d| < b \\ &y^{\cdot} + ay = a && \text{for } x^{\cdot} \leqslant 0,\ |x - b - d| < b \end{aligned} \tag{1.1}$$

$$\begin{aligned} &x^{\cdot\cdot} + x = -r\frac{x^{\cdot}}{|x^{\cdot}|} && \text{for } x^{\cdot} \geqslant 0,\ |x + b + d| > b \\ &y = 0 && \text{for } x^{\cdot} \leqslant 0,\ |x - b - d| > b \end{aligned} \tag{1.2}$$

Transition from (1.2) to (1.1) occurs for $x = -2b - d,\ x^{\cdot} > 0$ or $x = 2b + d,\ x^{\cdot} < 0$, and from (1.1) to (1.2) for $x = -d,\ x^{\cdot} > 0$ or $x = d,\ x^{\cdot} < 0$.

2. The point transformations. The phase space $x,\ y,\ z = x^{\cdot}$ consisting of part of the plane and the two three-dimensional domains matched to it is symmetric with respect to the y-axis.

We define the transformation S_1 as the mapping of a point of the half-line Γ_1 ($x = -2b - d,\ y = 0,\ z > 0$) with the coordinate $z = u$ along the trajectories of the upper-half-space into a point with the coordinate $z = v$ on the plane $x = -d$. The transformation S_2 is defined by virtue of the adopted idealization of contact device actuation [2]) as an instantaneous jump along the plane $x = -d$ onto the half-line Γ_2 ($x = -d$, $y = 0,\ z > 0$) of a representing point which has reached the plane $x = -d$.

The transformation S_3 is defined as the mapping of a point of the half-line Γ_2 with the coordinate $z = v_1$ into a point of the half-line $\overline{\Gamma}_1$ (symmetric to Γ_1 with respect to the y-axis). By virtue of the symmetry of the phase space of our problem we can identify the half-lines Γ_1 and $\overline{\Gamma}_1$.

Investigation of the decomposition of the phase space into trajectories reduces to the investigation of the point transformation $T = \overline{S_1S_2S_3}$ of the half-line Γ_1 into itself (the bar indicates mapping into a symmetric point following execution of the transformation $S_1S_2S_3$).

We have the following analytic expressions for the transformation S_1:

$$u = \frac{1}{\sin\tau}\left[2b\cos\tau + (r - d - 1)(1 - \cos\tau) + 2F(a, \tau) - F(2a, \tau)\right]$$

$$v = \frac{1}{\sin\tau}\left[2b - (r - d - 1)(1 - \cos\tau) - 2\Phi(a, \tau) + \Phi(2a, \tau)\right] \tag{2.1}$$

where

$$F(a, \tau) = \frac{1}{1 + a^2}\left(e^{-a\tau} - \cos\tau + a\sin\tau\right)$$

$$\Phi(a, \tau) = \frac{1 - e^{-a\tau}(\cos\tau + a\sin\tau)}{1 + a^2}$$

and τ is the transit time.

The surface $\{\alpha_1\}$ consisting of the plane

$$r = 2b + d \quad \text{for } b \leqslant b_0 \equiv 1 - \Phi(a, \pi) + {}^1/_2\,\Phi(2a, \pi)$$

and the surface

$$r = 1 + b + d - \Phi(a, \pi) + {}^1/_2\, \Phi(2a, \pi) \quad \text{for } b \geqslant b_0$$

isolate the domain in question in the parameter space a, b, d, r ; this domain is such that for all its points the half-line Γ_1 is mapped onto the plane $x = -d$ along the trajectories of the upper half-space (the parameter τ in expressions (2.1) varies in the range $0 < \tau \leqslant \tau_0 \leqslant \pi$).

We have the following analytic expression for the transformation S_3 :

$$[v_1^2 + (r-d)^2]^{1/2} - [u^2 + (r - 2b - d)^2]^{1/2} = 2r \tag{2.2}$$

The transformation S_3 is effected in the domain in question for

$$v_1 \geqslant v_0 \equiv 2\ [(r+b)\ (d+b)]^{1/2}$$

Here v_0 is the coordinate of the point on the half-line Γ_2 through which the trajectory of Eq. (1.2) tangent to the half-line $\bar{\Gamma}_1$ passes.

3. The singularities of the phase space. The parameter space. The surfaces $\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$ and $\{\alpha_4\}$ decompose the parameter space a, b, r ($d = \text{const} > 0$) into domains whose points correspond to various qualitative structures of the decomposition of the phase space into trajectories.

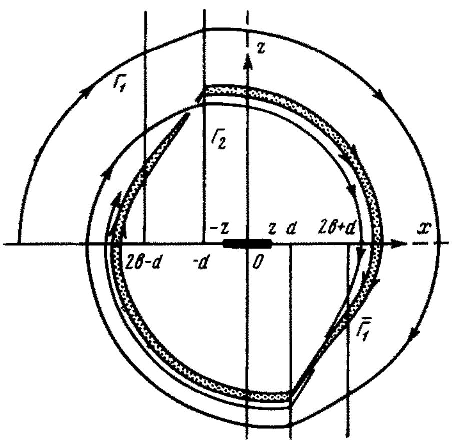

Fig. 1

For small values of the parameter r the point transformation T has a stable fixed point (a symmetric stable limit cycle exists in the phase space of the system). By increasing the parameter r with the remaining parameters held fixed it is possible to attain values of r for which the indicated limit cycle and some of the trajectories of the phase space lie on a doubly twisted strip (Fig. 1).

Further increases in the parameter r produce the bifurcation surface $\{\alpha_2\}$ defined by the conditions $dv/dv_1 = -1, \qquad v = v_1$

On passing through the surface $\{\alpha_2\}$ in the parameter space the fixed point of the transformation T experiences a change in stability; depending on the sign of some quantity $g_0 \neq 0$, it either generates two stable fixed points of the transformation T^2 (for $g_0 < 0$) or merges [2] with two unstable points of the transformation T^2 (for $g_0 > 0$). The two fixed points of transformation T^2 in the phase space of the system under consideration are associated with two nonsymmetric but symmetrically situated limit cycles.

The surface $\{\alpha_3\}$ is defined by the condition of passage of the nonsymetric limit cycle through the point $z = v_0$ of the half-line Γ_2. Passage through the surface $\{\alpha_3\}$ in the parameter space is accompanied by the appearance (generation from the boundary of the attraction domain of the rest segment $-r < x < r,\ y = 0,\ z = 0$) or by the disappearance (merging with the latter boundary) of two nonsymmetric limit cycles in the phase space.

The surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ intersect. Computations show that in the neighborhood of their lines of intersection on the surface $\{\alpha_2\}$ there exists a curve at whose points g_0

vanishes. Depending on the signs of the quantities [4] g_0 and h_0, the neighborhood of the fixed point of the transformation T can contain one or two pairs of fixed points of the transformation T^2 (two or four nonsymmetric limit cycles in the phase space). From the curve defined by the condition $g_0 = 0$ and situated on the surface $\{\alpha_2\}$ there emerges the bifyrcation surface $\{\alpha_4\}$ on which the two pairs of fixed points of the transformation merge and vanish [4] (the points of the surface $\{\alpha_4\}$ are associated with a phase space with two semistable nonsymmetric limit cycles).

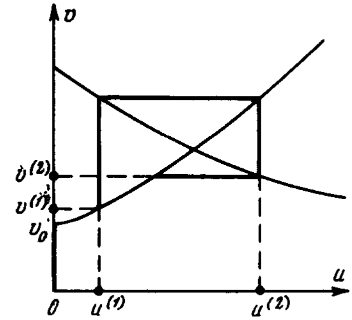

Fig. 2

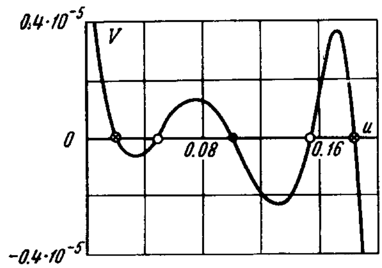

Fig. 3

The Laméray diagram constructed in the neighborhood of the point of intersection of the curves $v = v(u)$ (2.1) and $v_1 = v_1(u)$ (2.2) (the fixed point of the transformation T) for parameter values taken from the neighborhood of the surface $\{\alpha_2\}$ is shown in Fig.2. The fixed points of the transformation T^2 can be investigated by means of a "second" Laméray diagram [2]. As we know, the fixed point of the transformation T^2 corresponds to the existence of a rectangle on the Laméray diagram. We shall presently describe a method for computing a function whose zeros correspond to the existence of rectangles on the diagram, and therefore to the existence of fixed points of the transformation T^2. We denote the abscissa of the starting point by $u^{(1)}$ (its ordinate is $v^{(1)} = S^{-1}\bar{u}^{(1)}$) and the abscissa of the end point by $u^{(2)}$ (its ordinate is $v^{(2)} = \overline{S_1S_2S_3}\, S_1S_2u^{(1)}$). The required function is $V(u) = v^{(2)} - v^{(1)}$.

The surface $\{\alpha_4\}$ is defined by the parameter values for which the function $V(u)$ touches the axis of abscissas twice. We determined this surface approximately on a BESM-3M computer for the parameter values $a = 2$, $d = 0.2$. Here are the resulting values b and r of the coordinates of the point A of the surface $\{\alpha_2\}$ defined by the condition $g_0 = 0$ of the point B (the point of intersection of the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$, and of the point C (the point of intersection of the surfaces $\{\alpha_3\}$ and $\{\alpha_4\}$):

points	A	B	C
$b =$	0.12002	0.12065	0.121225
$r =$	0.10609	0.10656	0.10698

The surfaces $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_4\}$ isolate a domain in the parameter space which is associated with a phase space with five limit cycles, namely, one stable symmetric, two stable nonsymmetric, and two unstable nonsymmetric cycles. In the neighborhood of the point A there is a bifurcation [4] which corresponds to the case $h_0 < 0$. Figure 3 shows the curve of the function $V(u)$ constructed for the parameter values $b = 0.12060$, $r = 0.10652$ taken from this domain. The zeros of the function indicated by the blank circles in the figure correspond to two unstable nonsymmetric limit cycles; the zeros indicated by the crossed circles correspond to the two stable nonsymmetric limit cycles;

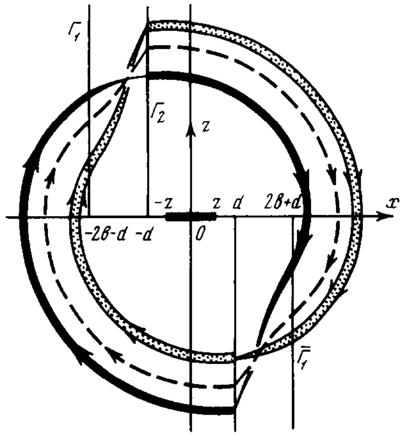

Fig. 4

the zero of the function indicated by the dot corresponds to the stable symmetric limit cycle.

For the domain situated between the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ to the left of the point B the phase space contains a symmetric unstable and two nonsymmetric stable limit cycles (Fig. 4). Figure 5 shows the corresponding curve of the function $V(u)$ constructed for the parameter values $b = 0.12040$, $r = 0.106375$ taken from the indicated domain.

For the domain situated between the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ to the right of the point B the phase space contains a symmetric stable and two nonsymmetric unstable limit cycles. Figure 6 shows the curve of the function $V(u)$ constructed for the parameter values $b = 0.12120$, $r = 0.10696$ taken from the indicated domain.

For the domain situated above the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ near these surfaces in the phase space there exists a symmetric unstable limit cycle which vanishes with further increases in the parameter r. The unstable limit cycle lies on a doubly twisted strip and does not divide the phase space into parts from which the representing points travel towards various attracting elements.

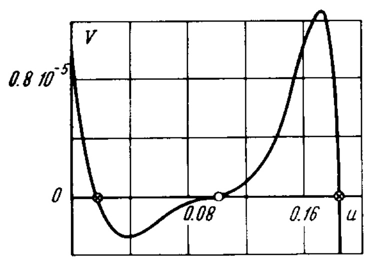

Fig. 5

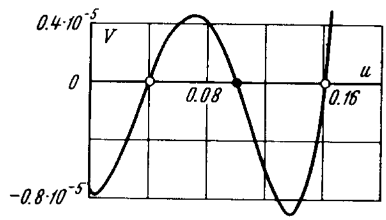

Fig. 6

Here are the values of b, r_2 and r_3 which correspond to the intersections of the surfaces $\{\alpha_2\}$ and $\{\alpha_3\}$ by the planes $a = 2$ and $d = 0.2$.

b =	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60
r_2 =	0.0442	0.0903	0.0127	0.1551	0.1949	0.2212	0.2398	0.2536
r_3 =	0.0459	0.0909	0.0126	0.1536	0.1929	0.2192	0.2380	0.2520

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DETERMINING THE COEFFICIENTS OF THE LEGENDRE POLYNOMIAL EXPANSION OF THE EARTH'S GRAVITATIONAL POTENTIAL

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The solution of the Stokes problem [1–4] is used to find exact expressions for the coefficients of the Legendre polynomial expansion of the potential of the Earth's regularized gravitational field with the Clairaut ellipsoid taken as the equipotential surface.

1. The solution of the Stokes problem with the Clairaut ellipsoid taken as the equipotential surface of the Earth's gravitational field [4] yields the following expression for the potential V of this field [1–3] in the Earth-centered orthogonal coordinate system $Oxyz$ (the origin O of this system coincides with the Earth's center; its z-axis is directed along the Earth's axis of rotation):

$$V(x, y, z) = -AP(x^2 + y^2) - BQz^2 + CR \tag{1.1}$$

Here

$$P = \operatorname{arc\,tg} \varepsilon' - \frac{\varepsilon'}{1+\varepsilon'^2}, \qquad Q = \varepsilon' - \operatorname{arc\,tg} \varepsilon', \qquad R = \operatorname{arc\,tg} \varepsilon' \tag{1.2}$$

where ε' is the second eccentricity of the ellipsoid which is confocal with the Clairaut ellipsoid and passes through the point at which the potential is being determined; A, B and C are constants.

The quantity ε' is given by the equation

$$\varepsilon' = [(a^2 - b^2) / (b^2 + \nu)]^{1/2} \tag{1.3}$$

where a and b are the major and minor semiaxes of the Clairaut ellipsoid and ν is the positive root of the equation

$$\frac{x^2 + y^2}{a^2 + \nu} + \frac{z^2}{b^2 + \nu} = 1 \tag{1.4}$$

The constants A, B and C appearing in formula (1.1) can be determined from the relations

$$A = \frac{u^2(1+\varepsilon^2)}{2[(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon - 3\varepsilon]}, \qquad B = 2A$$
$$C = \frac{a^2}{\varepsilon}\left\{\frac{g_e + u^2 a}{a} + \frac{u^2(1+\varepsilon^2)(\varepsilon - \operatorname{arc\,tg}\varepsilon)}{(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon - 3\varepsilon}\right\} \tag{1.5}$$

Here $\varepsilon = (a^2 - b^2)^{1/2} / b$ is the second eccentricity of the Clairaut ellipsoid, u is the Earth's angular velocity, and g_e is the gravitational acceleration at the equator.

Relations (1.1)–(1.5) yield an implicit expression for the potential $V(x, y, z)$. This expression is inconvenient for practical computations, which is why the gravitational